Episode 2

Motion of Particles in Cartesian Coordinates

ENGN0040: Dynamics and Vibrations
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Topics for todays class

Describing motion of particles

1. Definition of a particle
2. Position-velocity-acceleration relations for a particle
3. Inertial reference frames
4. Review of some calculus
5. Analyzing straight line motion of particles
6. Using MATLAB to integrate/differentiate the measured position/velocity/acceleration of a particle

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2 Describing motion of particles

2.1 Definition

A particle is a point mass at some position in space

Properties: (a) Mass
(b) Position (velocity, acceleration)
No shape or orientation

Examples: Satellite in space (for orbit)
Atom in an MD simulation

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2 Describing motion of particles

2.1 Definition

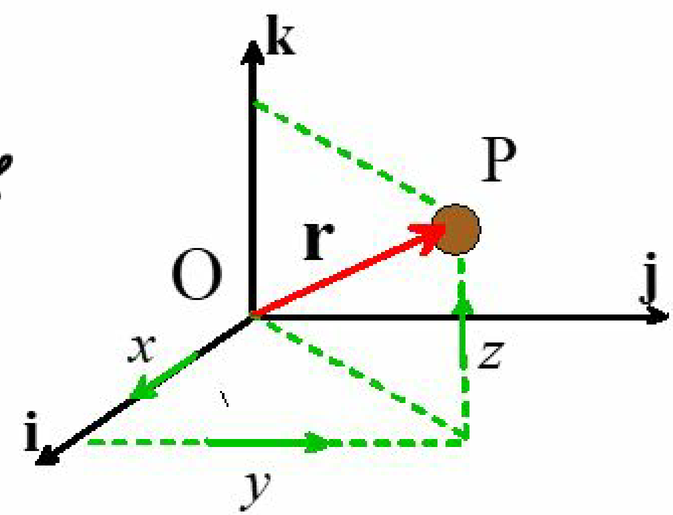
A particle is a point mass @ some position in space

Properties: Mass
Position (Velocity, accel)
No shape or orientation

Examples: Satellite in space (for orbit)
Atom in an MD simulation

2.2 Position - Velocity - Acceleration formulas (Cartesian coords)

2.2.1 Position



$\{O, \underline{i}, \underline{j}, \underline{k}\}$ is a Cartesian basis

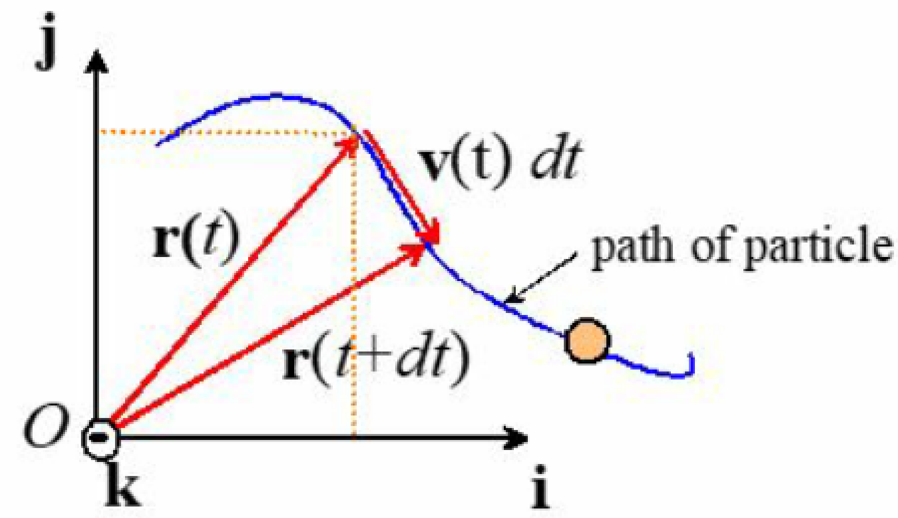
$$\underline{r} = x(t)\,\underline{i} + y(t)\,\underline{j} + z(t)\,\underline{k}$$

2.2.2 The "Inertial Basis"

To use Newton:
(a) O must not accelerate
(b) $\{\underline{i}, \underline{j}, \underline{k}\}$ must not rotate

This is an approximation! Use judgement!

2.2.3 Velocity



$$\underline{V} = V_x \underline{i} + V_y \underline{j} + V_z \underline{k}$$

Speed $$V = |\underline{V}| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

Direction is tangent to path

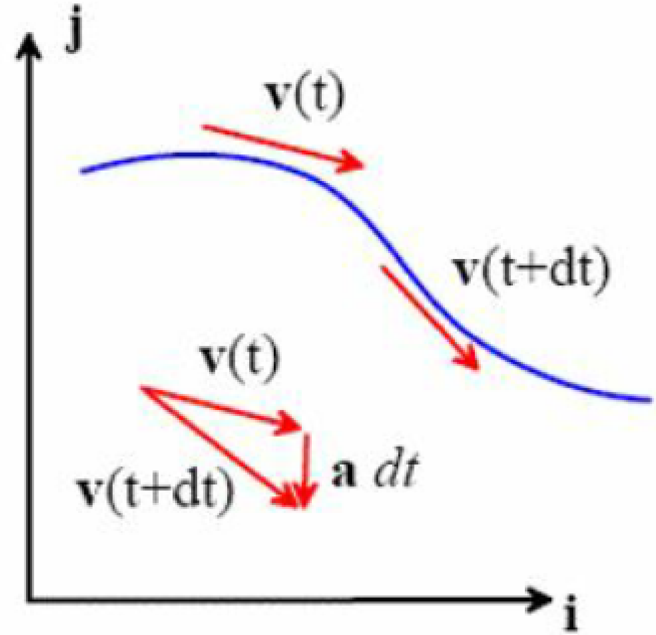
Definition $$\underline{V} = \frac{d\underline{r}}{dt}$$

$$= \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k}$$

Hence $V_x = \dfrac{dx}{dt}$ $\quad V_y = \dfrac{dy}{dt}$ $\quad V_z = \dfrac{dz}{dt}$

2.2.4 Acceleration

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$



Definition

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{dV_x}{dt}\underline{i} + \frac{dV_y}{dt}\underline{j} + \frac{dV_z}{dt}\underline{k}$$

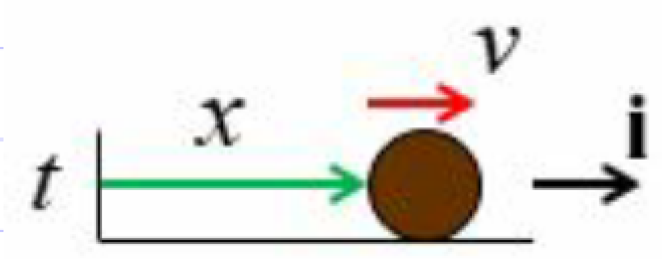
$$= \frac{d^2x}{dt^2}\underline{i} + \frac{d^2y}{dt^2}\underline{j} + \frac{d^2z}{dt^2}\underline{k}$$

Hence

$$a_x = \frac{dV_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y = \frac{dV_y}{dt} = \frac{d^2y}{dt^2} \qquad a_z = \frac{dV_z}{dt} = \frac{d^2z}{dt^2}$$

2.2.5 Special Case : 1-D motion

$$V = \frac{dx}{dt} \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$



Formula for a in terms of x

Suppose we know v as a function of x (eg $v = x^2$)

Calculate a using chain rule

$$\boxed{a = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}}$$

$$\left(\text{For } v = x^2 \quad a = 2x \cdot v = 2x^3 \right)$$

Particle motion problems we need to solve

(a) Given (x, y, z) in terms of time
Find (V_x, V_y, V_z) (a_x, a_y, a_z)

Regular calculus: MA0010, or 90, etc

(b) Given (a_x, a_y, a_z) in terms of t,
and sometimes (V_x, V_y, V_z), (x, y, z)
Find (V_x, V_y, V_z) (x, y, z)

To do this, we will need to solve
"differential equations" - eg $\frac{d^2x}{dt^2} + x = 0$

Solve by hand, or with MATLAB

Background

Common dynamics problem:

Given: (i) Acceleration $a = f(x, v, t)$

(ii) Speed v and position x at time $t = 0$

Find: speed v and distance traveled x for $t > 0$

Approach: Solve the differential equations

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d^2x}{dt^2} = f(x, v, t)$$

Some simple cases can be solved by hand.

(For harder equations we have to use MATLAB)

Some equations can be solved using *separation of variables*

Separation of variables

Three cases:

(1) Acceleration is a known function of time $a = f(t)$

Example: Rocket in space, with thrust that decreases with time (for $0 < t < T$)

$$a(t) = F_0(1 - t/T)/m$$

(2) Acceleration depends on speed (and time) $a = f(v)g(t)$

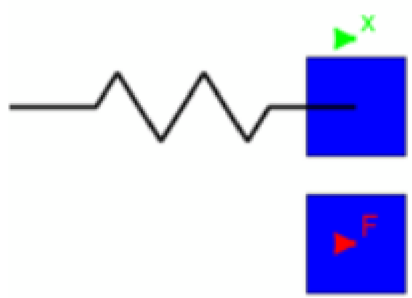
Example: dust particle dropping vertically with air resistance

$$a(v) = g - cv/m$$

(3) Acceleration depends on position (and speed) $a = f(x)g(v)$

Example: mass on a spring

$$a(x) = -kx/m$$



Separation of variables: Case 1

Acceleration is a known function of time $a = f(t)$

$$\frac{dv}{dt} = f(t) \qquad \frac{dx}{dt} = v \qquad \text{Initial condition:} \quad x(t=0) = x_0 \qquad v(t=0) = v_0$$

Calculating v

Step 1: 'Separate variables' $dv = f(t)dt$

Step 2: Integrate both sides $\int_{v_0}^{v} dv = \int_{0}^{t} f(t)dt$

Example $a = F_0(1 - t/T)/m \qquad 0 < t < T$

This means substitute the limits

Step 1 $dv = \{F_0(1 - t/T)/m\}dt$

Step 2 $\int_{v_0}^{v} dv = \int_{0}^{t} \{F_0(1 - t/T)/m\}dt \quad [v]_{v_0}^{v} = \left[F_0\left\{t - t^2/(2T)\right\}/m\right]_0^t$

$$\Rightarrow v = v_0 + F_0\left\{t - t^2/(2T)\right\}/m$$

Separation of variables: Case 1

Acceleration is a known function of time $a = f(t)$

$$\frac{dv}{dt} = f(t) \qquad \frac{dx}{dt} = v \qquad \text{Initial condition: } \; x(t=0) = x_0 \qquad v(t=0) = v_0$$

Calculating x

Step 1: 'Separate variables' $dx = v(t)dt$

Step 2: Integrate both sides $\int_{x_0}^{x} dx = \int_{0}^{t} v(t)dt$

Example $a = F_0(1 - t/T)/m \qquad v = v_0 + F_0\left\{t - t^2/(2T)\right\}/m \qquad 0 < t < T$

Step 1 $dx = \left(v_0 + F_0\left\{t - t^2/(2T)\right\}/m\right)dt$

Step 2 $\int_0^x dx = \int_0^t \left(v_0 + F_0\left\{t - t^2/(2T)\right\}/m\right)dt$

$$\Rightarrow \; [x]_0^x = \left[v_0 t + F_0\left\{t^2/2 - t^3/(6T)\right\}/m\right]_0^t$$

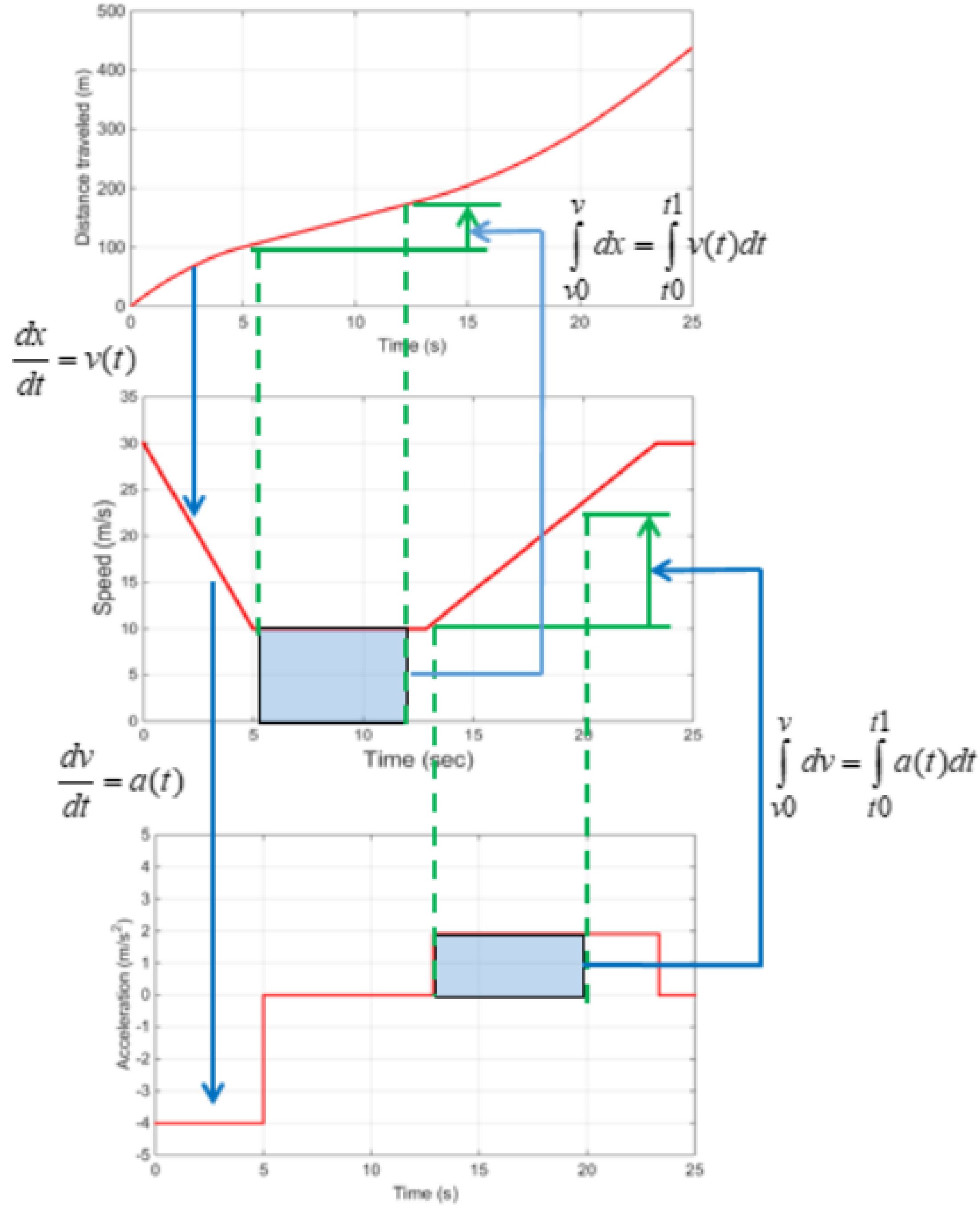
$$\Rightarrow x = v_0 t + F_0\left\{t^2/2 - t^3/(6T)\right\}/m$$

Separation of variables: Case 1

Graphical Method for
a, v, x all functions of time

- Speed is the slope of the distance-v-time curve
- Distance is the area under the speed-v-time curve

- Acceleration is the slope of the speed-v-time curve
- Speed is the area under the acceleration-v-time curve



Separation of variables: Case 2

Acceleration depends on speed (and time) $a = f(v)g(t)$

$$\frac{dv}{dt} = f(v)g(t) \qquad \frac{dx}{dt} = v \qquad \text{Initial condition:} \quad x(t=0) = x_0 \qquad v(t=0) = v_0$$

Calculating v

Step 1: 'Separate variables' $\frac{dv}{f(v)} = g(t)dt$

Step 2: Integrate both sides $\int_{v_0}^{v} \frac{dv}{f(v)} = \int_0^t g(t)dt$

Example $a(v) = g - cv/m$ $\longleftarrow$ $f(v) = g - cv/m$ $\qquad g(t) = 1$

Step 1 $\frac{dv}{g - cv/m} = dt$

Natural log (ln)

Step 2 $\int_{v_0}^{v} \frac{dv}{g - cv/m} = \int_0^t dt \quad \Rightarrow \quad \left[-\frac{m}{c}\log(g - cv/m)\right]_{v_0}^{v} = t$

$$\Rightarrow -\frac{m}{c}\log\left(\frac{g - cv/m}{g - cv_0/m}\right) = t \quad \Rightarrow \quad v = \frac{mg}{c} - \left(\frac{mg}{c} - v_0\right)\exp(-ct/m)$$

Separation of variables: Case 2

Acceleration depends on speed (and time) $a = f(v)g(t)$

$$\frac{dv}{dt} = f(v)g(t) \qquad \frac{dx}{dt} = v \qquad \text{Initial condition:} \quad x(t=0) = x_0 \qquad v(t=0) = v_0$$

Calculating x

Step 1: 'Separate variables' $dx = v(t)dt$

Step 2: Integrate both sides $\int_{x_0}^{x} dx = \int_{0}^{t} v(t)dt$

Example $a(v) = g - cv/m \qquad v = mg/c - (mg/c - v_0)\exp(-ct/m)$

Step 1 $dx = \left(mg/c - (mg/c - v_0)\exp(-ct/m)\right)dt$

Step 2 $\int_{x_0}^{x} dx = \int_{0}^{t}\left(mg/c - (mg/c - v_0)\exp(-ct/m)\right)dt$

$$\Rightarrow [x]_{x_0}^{x} = \left[mgt/c + (m/c)(mg/c - v_0)\exp(-ct/m)\right]_0^t$$

$$\Rightarrow x = x_0 + mgt/c + (m/c)(mg/c - v_0)\{\exp(-ct/m) - 1\}$$

Separation of variables: Case 3

Acceleration depends on position (and speed) $a = f(x)g(v)$

$$\frac{dv}{dt} = f(x)g(v) \qquad \frac{dx}{dt} = v \qquad \text{Initial condition:} \quad x(t=0) = x_0 \qquad v(t=0) = v_0$$

Calculating v

Step 1: Rewrite acceleration in terms of x $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v = f(x)g(v)$

Step 2: 'Separate variables' $\frac{vdv}{g(v)} = f(x)dx$

Step 3: Integrate both sides $\int_{v_0}^{v} \frac{vdv}{g(v)} = \int_{0}^{x} f(x)dx$

Example $a = -kx/m$

Steps 1&2 $vdv = (-kx/m)dx$

Step 3 $\int_{v_0}^{v} vdv = \int_{x_0}^{x} (-kx/m)dt \Rightarrow \left[v^2/2\right]_{v_0}^{v} = \left[-kx^2/(2m)\right]_{x_0}^{x}$

$$\Rightarrow \quad v = \sqrt{v_0^2 + kx_0^2/m - kx^2/m}$$

Separation of variables: Case 3

Acceleration depends on position (and speed) $a = f(v)g(x)$

$$\frac{dv}{dt} = f(x)g(v) \qquad \frac{dx}{dt} = v(x) \qquad \text{Initial condition:} \quad x(t=0) = x_0 \qquad v(t=0) = v_0$$

Calculating x

Step 1: 'Separate variables' $\quad \dfrac{dx}{v(x)} = dt$

Step 2: Integrate both sides $\quad \displaystyle\int_{x_0}^{x} \frac{dx}{v(x)} = \int_0^t dt$

Example $\quad a = -kx/m \qquad v = \sqrt{v_0^2 + kx_0^2/m - kx^2/m}$

Step 1 $\quad \dfrac{dx}{\sqrt{v_0^2 + kx_0^2/m - kx^2/m}} = dt$

Substitute $kx^2/(mv_0^2 + kx_0^2) = \sin^2(u)$

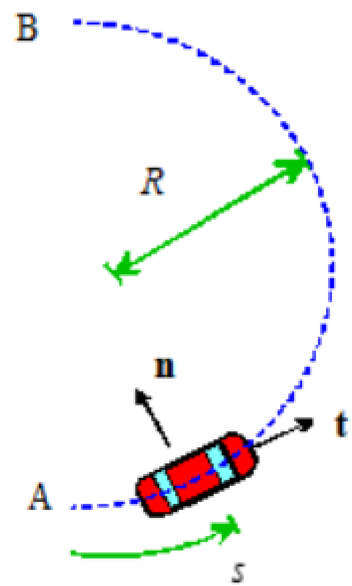
Step 2 $\quad \displaystyle\int_{x_0}^{x} \frac{dx}{\sqrt{v_0^2 + kx_0^2/m - kx^2/m}} = \int_0^t dt \Rightarrow \frac{1}{\sqrt{k/m}} \left[\sin^{-1}\left(\frac{x}{\sqrt{mv_0^2/k + x_0^2}} \right) \right]_{x_0}^{x} = t$

$$\Rightarrow x = \sqrt{mv_0^2/k + x_0^2} \sin\left(\sqrt{(k/m)}\, t + \sin^{-1}\left(\frac{x_0}{\sqrt{mv_0^2/k + x_0^2}} \right) \right)$$

Final note

We use the same calculus in many other applications

Example 1: Motion along a curved path



Differential equations

Replaces *x*

$$\frac{dV}{dt} = V\frac{dV}{ds} = a_t(s,V,t) \qquad \frac{ds}{dt} = V$$

Replaces *v*

Example 2: Rotation about a fixed axis

Differential equations

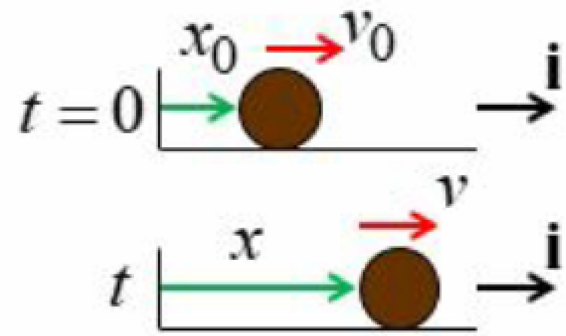
Replaces *x*

$$\frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta} = \alpha(\theta,\omega,t) \qquad \frac{d\theta}{dt} = \omega$$

Replaces *a* Replaces *v*

2.4 Example: Straight line motion with constant acceleration

A particle has constant acceleration $\mathbf{a} = a\mathbf{i}$
At time $t=0$ it has velocity $\mathbf{v} = v_0\mathbf{i}$ and position $\mathbf{r} = x_0\mathbf{i}$
Find $\mathbf{v}(t)$, $\mathbf{v}(x)$ and $\mathbf{r}(t)$



Use calculus formulas

$$\frac{dv}{dt} = a \Rightarrow \int_{v_0}^{v} dv = \int_0^t a\,dt \Rightarrow v - v_0 = at$$

$$\boxed{v(t) = v_0 + at}$$

$$v\frac{dv}{dx} = a \Rightarrow \int_{v_0}^{v} v\,dv = \int_{x_0}^{x} a\,dx \Rightarrow \left[\frac{1}{2}v^2\right]_{v_0}^{v} = a(x - x_0)$$

$$\Rightarrow \frac{1}{2}v^2 - \frac{1}{2}v_0^2 = a(x - x_0)$$

$$\Rightarrow \boxed{v(x) = \sqrt{v_0^2 + 2a(x - x_0)}}$$

$$\frac{dx}{dt} = V = V_0 + at \Rightarrow \int_{x_0}^{x} dx = \int_0^t (V_0 + at)\,dt$$

$$\Rightarrow \quad x - x_0 = V_0 t + \frac{1}{2} a t^2$$

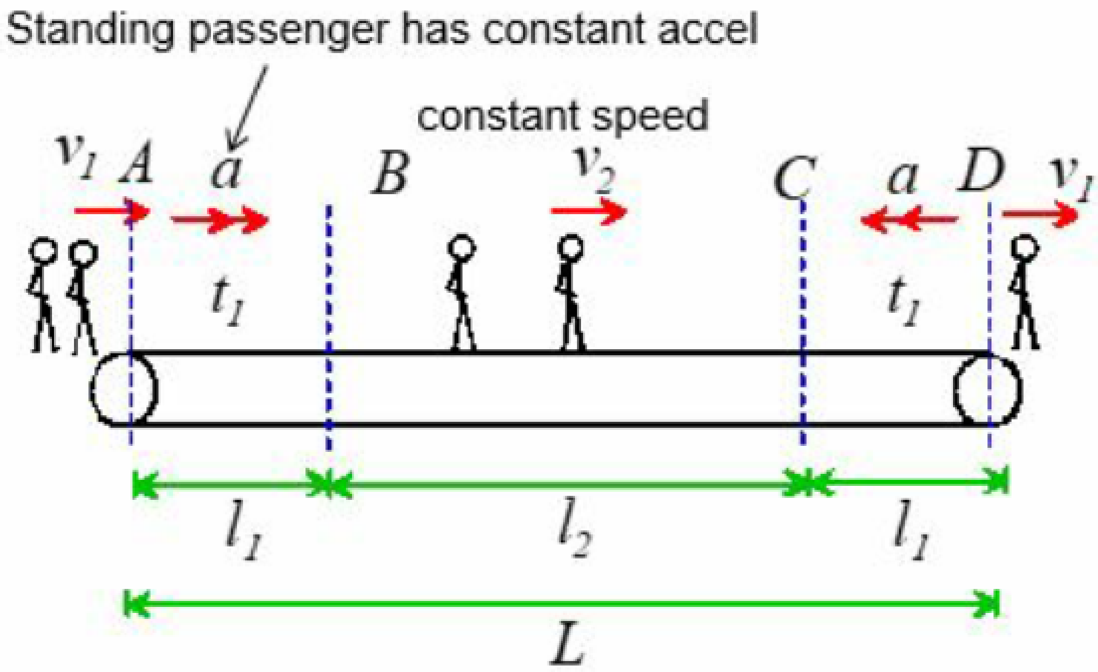
$$\boxed{x = x_0 + V_0 t + \frac{1}{2} a t^2}$$

"Constant acceleration formulas"

NB: Use these only if a is constant

Otherwise separate variables or use MATLAB

2.5 Example: Toronto high-speed walkway



Given information:

- Total length L= 912 ft,
- V1=125 ft/min
- V2=400 ft/min
- Travel time A->B = t1 approx 10 sec
- Travel time C->D = t1 approx 10 sec

Calculate:

(a) Acceleration a

(b) Time of travel T from A->D

(a) Use constant accel formulas between A & B

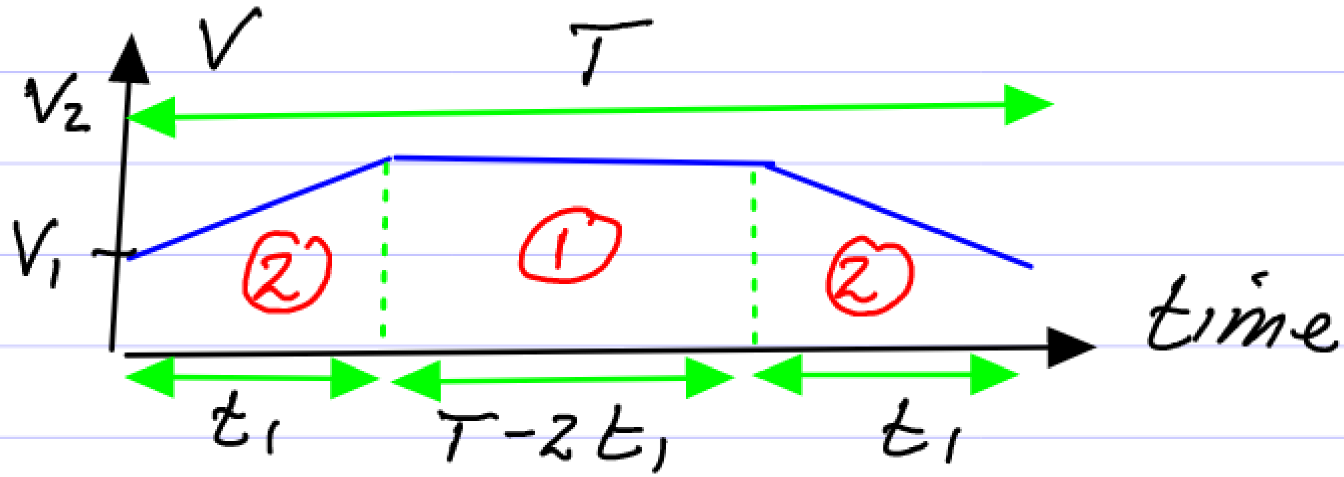
$$V_2 = V_1 + a t_1 \Rightarrow a = \frac{V_2 - V_1}{t_1} = \frac{400-125}{60 \times 10}$$

$$\Rightarrow \boxed{a = 0.46 \frac{ft}{s^2}}$$

(approx 0.015 g
g = 32 ft/s²)

(b) Use graphical method

Distance = area under graph



$$L = \underbrace{V_2(T-2t_1)}_{①} + \underbrace{2\left\{\frac{(V_1+V_2)}{2}\,t_1\right\}}_{②}$$

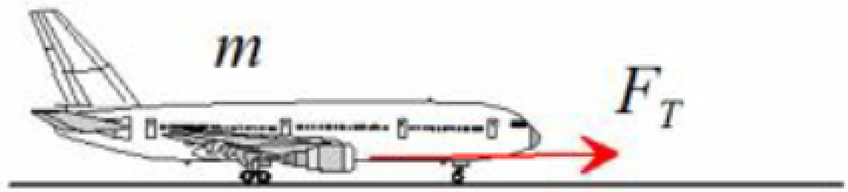
Solve for T: $T = \dfrac{L}{V_2} + 2t_1 - \dfrac{(V_1+V_2)\,t_1}{V_2}$

Substitute numbers $\boxed{T = 142.9\,s}$

(compare to $437\,s$ @ speed V_1)

2.5 Example: Straight line motion with variable acceleration

Aircraft starts from rest.



Acceleration $a = \frac{F_0}{m}\left(1 - \frac{v}{v_0}\right)$

Must reach speed v_{TO} to take off

1. Find a formula for speed as a function of time.
2. Find a formula for distance traveled as a function of time
3. Find a formula for the minimum length of runway required to takeoff

Use separation of variables

$$\frac{dV}{dt} = a = \frac{F_0}{m}\left\{1 - \frac{V}{V_0}\right\} \Rightarrow \int_0^V \frac{dV}{1 - V/V_0} = \int_0^t \frac{F_0}{m} dt$$

$$\Rightarrow \left[-V_0 \log\left(1 - V/V_0\right)\right]_0^V = \frac{F_0}{m} t$$

$$\Rightarrow \boxed{V = V_0\left(1 - \exp\left\{-\frac{F_0}{m V_0} t\right\}\right)}$$

(2) $$\frac{dx}{dt} = V = V_0\left(1 - \exp\left\{-\frac{F_0}{mV_0}t\right\}\right)$$

$$\Rightarrow \int_0^x dx = \int_0^t V_0\left(1 - \exp\left\{-\frac{F_0}{mV_0}t\right\}\right)dt$$

$$\Rightarrow x = \left[V_0 t + \frac{mV_0^2}{F_0}\exp\left\{-\frac{F_0}{mV_0}t\right\}\right]_0^t$$

$$\Rightarrow \boxed{x = V_0 t + \frac{mV_0^2}{F_0}\left(\exp\left\{-\frac{F_0}{mV_0}t\right\} - 1\right)}$$

(3) Notes: (a) Aircraft must reach V_{TO} before end of runway
(b) We can find time to reach V_{TO} from part (1); then find distance traveled from (2)

$$V_{TO} = V_0 \left(1 - \exp\left\{ -\frac{F_0}{mV_0} t \right\} \right)$$

Note: $\frac{mV_0^2}{F_0} \left(\exp\left\{ -\frac{F_0}{mV_0} t \right\} - 1 \right) = -\frac{mV_0 V_{TO}}{F_0}$

Now solve for t

$$t = -\frac{mV_0}{F_0} \log\left(1 - \frac{V_{TO}}{V_0} \right)$$

Finally $\boxed{x = -\frac{mV_0^2}{F_0} \log\left(1 - \frac{V_{TO}}{V_0} \right) - \frac{mV_0 V_{TO}}{F_0}}$

Note: $V_{TO} < V_0$ and $\log(\beta) < 0$ for $\beta < 1$
– first term is positive!

2.7 Example: Integrating/differentiating acceleration/position with MATLAB

Accelerometers on a quadcopter and a radio positioning system measure position, velocity, and acceleration at a series of successive times.

Data stored in 'comma separated value' (csv) file

Use MATLAB to read the file, and

(a) integrate the acceleration (a_x, a_y, a_z)

(b) Differentiate the position (x, y, z)

to find (v_x, v_y, v_z). Compare with measured velocity.

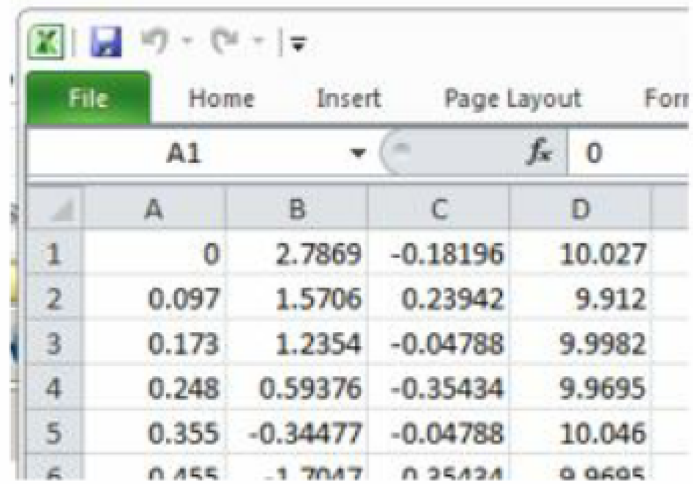
CSV files contain numbers separated by commas

1, 2, 3, 4

5, 6, 7, 8

etc…

Will open in EXCEL



	A	B	C	D
1	0	2.7869	-0.18196	10.027
2	0.097	1.5706	0.23942	9.912
3	0.173	1.2354	-0.04788	9.9982
4	0.248	0.59376	-0.35434	9.9695
5	0.355	-0.34477	-0.04788	10.046
6	0.455	-1.7047	0.35434	9.9695

Data columns in example file are $time, x, y, z, v_x, v_y, v_z, a_x, a_y, a_z$ (in SI units)

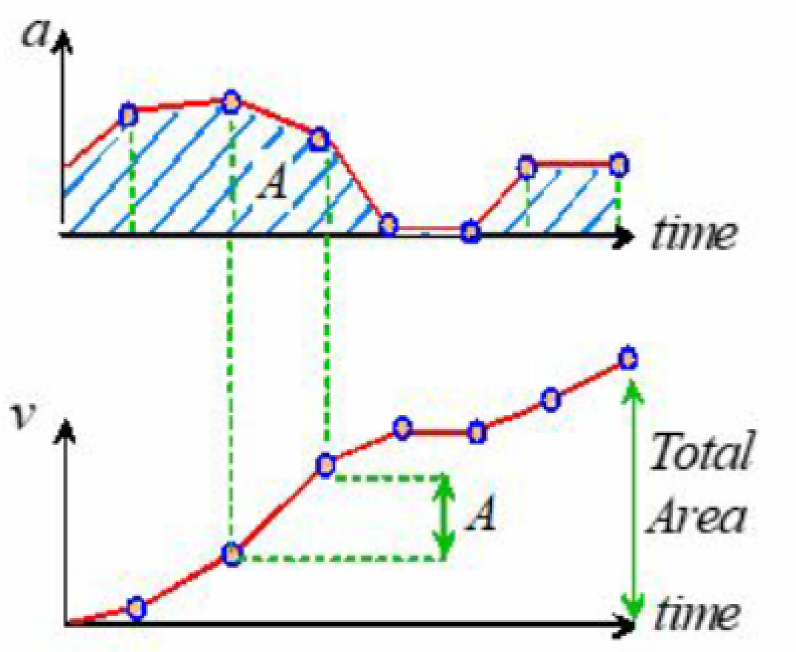
To read a csv file use

data = csvread ('filename.csv')

Matrix of data (rows, cols)

Integrating acceleration data with MATLAB

Algorithm: "Trapezoidal Integration"



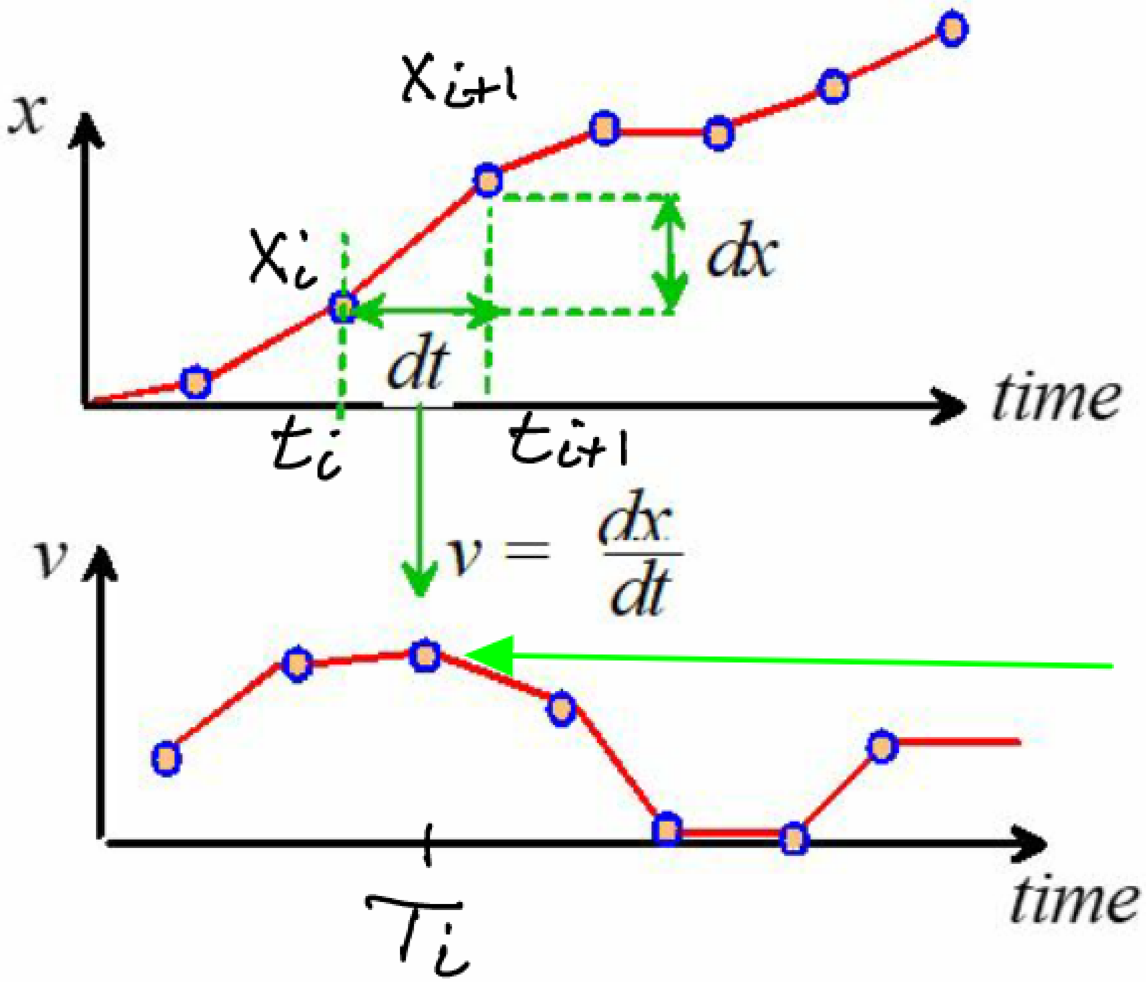
Acceleration data (plotted with 'plot')

Calculated velocity

MATLAB has a built-in function for this

$$\underbrace{V}_{\text{Vector of } v \text{ values}} = \text{cumtrapz}(\underbrace{\text{times}}_{\text{Vector of } t \text{ values}}, \underbrace{\text{accels}}_{\text{Vector of } a \text{ values}})$$

Differentiating positions with MATLAB



$$V_i = \frac{x_{i+1} - x_i}{t_{i+1} - t_i}$$

$$T_i = \frac{t_i + t_{i+1}}{2}$$

Find T_i, V_i for each i using a loop

```
function process_quadcopter_data
close all
data = csvread('demo_flight_data.csv');
time = data(:,1); x = data(:,2); y = data(:,3);z = data(:,4);
ax = data(:,8); vx = data(:,5);
plot3(x,y,z);
grid on;
title('Trajectory'); xlabel('x (m)'); ylabel('y (m)'); zlabel('z (m)');
figure
plot(time,ax);
grid on;
title('Acceleration'); xlabel('time (s)'); ylabel('a_x (m/s^2)');
v_integrated = cumtrapz(time,ax);
figure
plot(time,v_integrated,'Displayname','Integrated a_x')
hold on
plot(time,vx,'Displayname','v_x from quadcopter')
for i = 1:length(time) - 1
    T(i) = (time(i+1) + time(i))/2;
    v_differentiated(i) = (x(i+1)-x(i))/(time(i+1)-time(i));
end
%plot(T,v_differentiated);
v_smoothed = smooth(v_differentiated);
plot(T,v_smoothed,'Displayname','Differentiated x');
grid on;
title('Velocity Calculations'); xlabel('time (s)'); ylabel('v_x (m/s)');
legend(gca,'show');

end
```